

The Time Dependence of $B^0 \rightarrow f$ Decays

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Abstract

We calculate the time dependence of $B^0 \rightarrow f$ decays (and analogously $D^0 \rightarrow f$ decays) and use this result to obtain expressions for the time dependence of: an untagged sample ($B^0 \rightarrow f$ and $\bar{B}^0 \rightarrow f$ decays combined), a sample containing charge-conjugate final states ($B^0 \rightarrow f$ and $B^0 \rightarrow \bar{f}$ decays combined), and a sample containing all four decay modes together. For simplicity we assume CP violating effects are negligible.

The time dependence of $B^0 \rightarrow f$ decays, in which f is not a CP eigenstate, is not purely exponential due to the presence of B^0 - \bar{B}^0 mixing. This mixing arises due to either a mass difference Δm or a decay-width difference $\Delta\gamma$ between the mass eigenstates of the B^0 - \bar{B}^0 system. This note calculates an expression for the time dependence in the presence of such mixing. It is assumed that CP violation, if present, occurs at a negligible level. This assumption is well-motivated in the D^0 system, for which our results can be applied. In particular, our results can be applied to the decay $D^0 \rightarrow K^- \pi^+$ [1].

There are two strong eigenstates, B^0 and \bar{B}^0 , which are not eigenstates of the full Hamiltonian due to the weak interaction. Thus, they most generally evolve according to the Schrödinger equation:

$$i\frac{\partial}{\partial t} \begin{pmatrix} |B^0\rangle \\ |\bar{B}^0\rangle \end{pmatrix} = \left(\mathbf{M} - \frac{i}{2}\mathbf{\Gamma} \right) \begin{pmatrix} |B^0\rangle \\ |\bar{B}^0\rangle \end{pmatrix} \quad (1)$$

where the 2×2 Hermitian matrices \mathbf{M} and $\mathbf{\Gamma}$ represent transition amplitudes to virtual (off-mass-shell) and real (on-mass-shell) states, respectively. Diagonalizing $\mathbf{M} - (i/2)\mathbf{\Gamma}$ yields the physical states

$$\begin{aligned} |B_S\rangle &= p|B^0\rangle + q|\bar{B}^0\rangle \\ |B_L\rangle &= p|B^0\rangle - q|\bar{B}^0\rangle \end{aligned} \quad (2)$$

where

$$\frac{q}{p} = \sqrt{\frac{M_{12}^* - (i/2)\Gamma_{12}^*}{M_{12} - (i/2)\Gamma_{12}}} \approx \sqrt{\frac{M_{12}^*}{M_{12}}} = e^{i2\phi_M}. \quad (3)$$

In Eq. (2) we've used the notation B_L/B_S in analogy with the K_L/K_S system,¹ while in Eq. (3) we've made the good approximation $\Gamma_{12} \ll M_{12}$ [2]. The time evolution of the physical states is:

$$\begin{aligned} |B_S(t)\rangle &= |B_S\rangle e^{-(\Gamma_S/2+im_S)t} \\ |B_L(t)\rangle &= |B_L\rangle e^{-(\Gamma_L/2+im_L)t}. \end{aligned} \quad (4)$$

Inverting Eq. (2) gives:

$$\begin{aligned} |B^0\rangle &= \frac{1}{2p} (|B_S\rangle + |B_L\rangle) \\ |\bar{B}^0\rangle &= \frac{1}{2q} (|B_S\rangle - |B_L\rangle), \end{aligned} \quad (5)$$

and inserting the time dependences (4) into Eqs. (5) gives:

$$\begin{aligned} |B^0(t)\rangle &= \frac{1}{2p} \left\{ |B_S\rangle e^{-(\Gamma_S/2+im_S)t} + |B_L\rangle e^{-(\Gamma_L/2+im_L)t} \right\} \\ &= e^{-(\bar{\Gamma}/2+i\bar{m})t} \left\{ \cosh[(\Delta\gamma/4 + i\Delta m/2)t] |B^0\rangle + \left(\frac{q}{p}\right) \sinh[(\Delta\gamma/4 + i\Delta m/2)t] |\bar{B}^0\rangle \right\} \\ |\bar{B}^0(t)\rangle &= \frac{1}{2q} \left\{ |B_S\rangle e^{-(\Gamma_S/2+im_S)t} - |B_L\rangle e^{-(\Gamma_L/2+im_L)t} \right\} \\ &= e^{-(\bar{\Gamma}/2+i\bar{m})t} \left\{ \left(\frac{p}{q}\right) \sinh[(\Delta\gamma/4 + i\Delta m/2)t] |B^0\rangle + \cosh[(\Delta\gamma/4 + i\Delta m/2)t] |\bar{B}^0\rangle \right\}, \end{aligned}$$

where $\bar{m} \equiv (m_L + m_S)/2$, $\bar{\Gamma} \equiv (\Gamma_L + \Gamma_S)/2$, $\Delta m \equiv m_L - m_S$, and $\Delta\gamma \equiv \Gamma_L - \Gamma_S$. In the limit $\Delta\gamma \rightarrow 0$, one recovers the expressions which follow Eq. (5) in Ref. [3]. The states above lead to decay amplitudes:

$$\begin{aligned} \langle f|H|B^0(t)\rangle &= e^{-(\bar{\Gamma}/2+i\bar{m})t} \left\{ \cosh[(\Delta\gamma/4 + i\Delta m/2)t] \mathcal{A}_f + \left(\frac{q}{p}\right) \sinh[(\Delta\gamma/4 + i\Delta m/2)t] \bar{\mathcal{A}}_f \right\} \\ \langle f|H|\bar{B}^0(t)\rangle &= e^{-(\bar{\Gamma}/2+i\bar{m})t} \left\{ \left(\frac{p}{q}\right) \sinh[(\Delta\gamma/4 + i\Delta m/2)t] \mathcal{A}_f + \cosh[(\Delta\gamma/4 + i\Delta m/2)t] \bar{\mathcal{A}}_f \right\}, \end{aligned}$$

where we've defined the amplitudes for the pure B^0 and \bar{B}^0 states as:

$$\mathcal{A}_f \equiv \langle f|H|B^0\rangle \quad \bar{\mathcal{A}}_f \equiv \langle f|H|\bar{B}^0\rangle. \quad (6)$$

Squaring the decay amplitudes gives the decay rates:

$$\begin{aligned} |\langle f|H|B^0(t)\rangle|^2 &= |\mathcal{A}_f|^2 e^{-\bar{\Gamma}t} \left\{ |\cosh(\Delta\gamma/4 + i\Delta m/2)t|^2 + \right. \\ &\quad \left. |\lambda|^2 |\sinh(\Delta\gamma/4 + i\Delta m/2)t|^2 + \right. \\ &\quad \left. (\lambda^*) \cosh(\Delta\gamma/4 + i\Delta m/2)t [\sinh(\Delta\gamma/4 + i\Delta m/2)t]^* + c.c. \right\} \end{aligned} \quad (7)$$

¹Although $|\tau_{B_L} - \tau_{B_S}| \ll |\tau_{K_L} - \tau_{K_S}|$.

$$\begin{aligned}
|\langle f|H|\bar{B}^0(t)\rangle|^2 &= |\bar{\mathcal{A}}_f|^2 e^{-\bar{\Gamma}t} \left\{ |\cosh(\Delta\gamma/4 + i\Delta m/2)t|^2 + \right. \\
&\quad |\bar{\lambda}|^2 |\sinh(\Delta\gamma/4 + i\Delta m/2)t|^2 + \\
&\quad \left. (\bar{\lambda}^*) \cosh(\Delta\gamma/4 + i\Delta m/2)t [\sinh(\Delta\gamma/4 + i\Delta m/2)t]^* + c.c. \right\},
\end{aligned} \tag{8}$$

where we've defined the parameters $\lambda \equiv (q/p)(\bar{\mathcal{A}}_f/\mathcal{A}_f)$ and $\bar{\lambda} \equiv (p/q)(\mathcal{A}_f/\bar{\mathcal{A}}_f)$. To evaluate these expressions, note that:

$$\begin{aligned}
|\cosh(\Delta\gamma/4 + i\Delta m/2)t|^2 &= |\cosh(\Delta\gamma/4)t \cos(\Delta m/2)t + i \sinh(\Delta\gamma/4)t \sin(\Delta m/2)t|^2 \\
&= \cosh^2(\Delta\gamma/4)t \cos^2(\Delta m/2)t + \sinh^2(\Delta\gamma/4)t \sin^2(\Delta m/2)t \\
&= \cosh^2(\Delta\gamma/4)t - \sin^2(\Delta m/2)t \\
&= \left(\frac{1}{2}\right) [\cosh(\Delta\gamma/2)t + \cos(\Delta m)t].
\end{aligned} \tag{9}$$

Similarly,

$$\begin{aligned}
|\sinh(\Delta\gamma/4 + i\Delta m/2)t|^2 &= |\sinh(\Delta\gamma/4)t \cos(\Delta m/2)t + i \cosh(\Delta\gamma/4)t \sin(\Delta m/2)t|^2 \\
&= \sinh^2(\Delta\gamma/4)t \cos^2(\Delta m/2)t + \cosh^2(\Delta\gamma/4)t \sin^2(\Delta m/2)t \\
&= \cosh^2(\Delta\gamma/4)t - \cos^2(\Delta m/2)t \\
&= \left(\frac{1}{2}\right) [\cosh(\Delta\gamma/2)t - \cos(\Delta m)t].
\end{aligned} \tag{10}$$

Finally,

$$\begin{aligned}
&\cosh(\Delta\gamma/4 + i\Delta m/2)t [\sinh(\Delta\gamma/4 + i\Delta m/2)t]^* = \\
&\quad [\cosh(\Delta\gamma/4)t \cos(\Delta m/2)t + i \sinh(\Delta\gamma/4)t \sin(\Delta m/2)t] \times \\
&\quad [\sinh(\Delta\gamma/4)t \cos(\Delta m/2)t - i \cosh(\Delta\gamma/4)t \sin(\Delta m/2)t] \\
&= \left(\frac{1}{2}\right) [\sinh(\Delta\gamma/2)t \cos^2(\Delta m/2)t + \sinh(\Delta\gamma/2)t \sin^2(\Delta m/2)t + \\
&\quad i \sinh^2(\Delta\gamma/4)t \sin(\Delta m)t - i \cosh^2(\Delta\gamma/4)t \sin(\Delta m)t] \\
&= \left(\frac{1}{2}\right) [\sinh(\Delta\gamma/2)t - i \sin(\Delta m)t].
\end{aligned} \tag{11}$$

Inserting Eqs. (9), (10), and (11) into Eqs. (7) and (8) gives:

$$\begin{aligned}
|\langle f|H|B^0(t)\rangle|^2 &= \left(\frac{|\mathcal{A}_f|^2}{2}\right) e^{-\bar{\Gamma}t} \left\{ [\cosh(\Delta\gamma/2)t + \cos(\Delta m)t] + \right. \\
&\quad |\lambda|^2 [\cosh(\Delta\gamma/2)t - \cos(\Delta m)t] + \\
&\quad \left. (\lambda^*) [\sinh(\Delta\gamma/2)t - i \sin(\Delta m)t] + \right.
\end{aligned}$$

$$\begin{aligned}
& (\lambda) [\sinh(\Delta\gamma/2)t + i \sin(\Delta m)t] \Big\} \\
= & \left(\frac{|\mathcal{A}_f|^2}{2} \right) e^{-\bar{\Gamma}t} \Big\{ (1 + |\lambda|^2) \cosh(\Delta\gamma/2)t + (1 - |\lambda|^2) \cos(\Delta m)t + \\
& (\lambda + \lambda^*) \sinh(\Delta\gamma/2)t + i(\lambda - \lambda^*) \sin(\Delta m)t \Big\} \\
& (12)
\end{aligned}$$

$$\begin{aligned}
|\langle f|H|\bar{B}^0(t)\rangle|^2 &= \left(\frac{|\bar{\mathcal{A}}_f|^2}{2} \right) e^{-\bar{\Gamma}t} \Big\{ [\cosh(\Delta\gamma/2)t + \cos(\Delta m)t] + \\
& |\bar{\lambda}|^2 [\cosh(\Delta\gamma/2)t - \cos(\Delta m)t] + \\
& (\bar{\lambda}^*) [\sinh(\Delta\gamma/2)t - i \sin(\Delta m)t] + \\
& (\bar{\lambda}) [\sinh(\Delta\gamma/2)t + i \sin(\Delta m)t] \Big\} \\
= & \left(\frac{|\bar{\mathcal{A}}_f|^2}{2} \right) e^{-\bar{\Gamma}t} \Big\{ (1 + |\bar{\lambda}|^2) \cosh(\Delta\gamma/2)t + (1 - |\bar{\lambda}|^2) \cos(\Delta m)t + \\
& (\bar{\lambda} + \bar{\lambda}^*) \sinh(\Delta\gamma/2)t + i(\bar{\lambda} - \bar{\lambda}^*) \sin(\Delta m)t \Big\}. \\
& (13)
\end{aligned}$$

If the lifetime distribution for a final state f is constructed from a sample of *untagged* decays, and there are equal numbers of B^0 and \bar{B}^0 mesons produced, then the time dependence of the decays will be the sum of Eqs. (12) and (13). If $|\bar{\mathcal{A}}_f| \ll |\mathcal{A}_f|$ (e.g., if $B^0 \rightarrow f$ is Cabibbo-favored and $\bar{B}^0 \rightarrow f$ is doubly-Cabibbo-suppressed), then one would measure:

$$\begin{aligned}
\frac{dN_{(B^0+\bar{B}^0)\rightarrow f}}{dt} &\approx \left(\frac{1}{2} \right) e^{-\bar{\Gamma}t} \Big\{ 2|\mathcal{A}_f|^2 \cosh(\Delta\gamma/2)t + \\
& \mathcal{A}_f^* \bar{\mathcal{A}}_f [(q/p) + (p/q)^*] \sinh(\Delta\gamma/2)t + \mathcal{A}_f \bar{\mathcal{A}}_f^* [(q/p)^* + (p/q)] \sinh(\Delta\gamma/2)t + \\
& \mathcal{A}_f^* \bar{\mathcal{A}}_f [(q/p) - (p/q)^*] i \sin(\Delta m)t + \mathcal{A}_f \bar{\mathcal{A}}_f^* [(p/q) - (q/p)^*] i \sin(\Delta m)t \Big\} \\
= & \left(\frac{1}{2} \right) e^{-\bar{\Gamma}t} \Big\{ 2|\mathcal{A}_f|^2 \cosh(\Delta\gamma/2)t + [\mathcal{A}_f^* \bar{\mathcal{A}}_f (2q/p) + \mathcal{A}_f \bar{\mathcal{A}}_f^* (2q/p)^*] \sinh(\Delta\gamma/2)t \Big\} \\
= & |\mathcal{A}_f|^2 e^{-\bar{\Gamma}t} \Big\{ \cosh(\Delta\gamma/2)t + (\lambda + \lambda^*) \sinh(\Delta\gamma/2)t \Big\}, \\
& (14)
\end{aligned}$$

where we've used the fact that $|q/p|^2 = 1$. This result [and Eq. (15) below] was previously obtained by Dunietz [4].

If the lifetime distribution includes final states f and \bar{f} combined together, then we must also consider the decay rates $|\langle \bar{f}|H|\bar{B}^0(t)\rangle|^2$ and $|\langle \bar{f}|H|B^0(t)\rangle|^2$. These rates are equivalent to Eqs. (12) and (13), respectively, with the interchange $q/p \leftrightarrow p/q$ (since $|\mathcal{A}_f| = |\bar{\mathcal{A}}_{\bar{f}}|$ and $|\bar{\mathcal{A}}_f| = |\mathcal{A}_{\bar{f}}|$ by CP conservation). The analog of Eq. (14) for the final state \bar{f} is then:

$$\frac{dN_{(B^0+\bar{B}^0)\rightarrow \bar{f}}}{dt} \approx |\mathcal{A}_f|^2 e^{-\bar{\Gamma}t} \Big\{ \cosh(\Delta\gamma/2)t + (\kappa + \kappa^*) \sinh(\Delta\gamma/2)t \Big\}, \quad (15)$$

where $\kappa \equiv (p/q)(\mathcal{A}_{\bar{f}}/\bar{\mathcal{A}}_{\bar{f}}) = (q/p)^*(\bar{\mathcal{A}}_f/\mathcal{A}_f)e^{i\theta}$. Thus,

$$\frac{dN_{(B^0+\bar{B}^0)\rightarrow(f+\bar{f})}}{dt} \approx 2|\mathcal{A}_f|^2 e^{-\bar{\Gamma}t} \left\{ \cosh(\Delta\gamma/2)t + \text{Re}(\lambda + \kappa) \sinh(\Delta\gamma/2)t \right\}. \quad (16)$$

For this expression to be valid the f and \bar{f} final states must have equal acceptances, or else the decays must be corrected for acceptance before the lifetime distributions are combined.

If the B^0 and \bar{B}^0 mesons are produced in a fixed-target experiment, then their yields will usually be different due to there being unequal numbers of d and \bar{d} quarks in the initial state. In this case Eq. (14) does not apply, and it is more convenient to consider B^0 and \bar{B}^0 samples separately. Such experiments typically combine together the final states f and \bar{f} (when not studying CP asymmetries), and in this case:

$$\begin{aligned} \frac{dN_{B^0\rightarrow(f+\bar{f})}}{dt} = |\mathcal{A}_f|^2 e^{-\bar{\Gamma}t} \left\{ (1 + |\lambda|^2) \cosh(\Delta\gamma/2)t + \right. \\ \left. \text{Re}(\lambda + \kappa^*) \sinh(\Delta\gamma/2)t - \text{Im}(\lambda + \kappa^*) \sin(\Delta m)t \right\} \end{aligned} \quad (17)$$

$$\begin{aligned} \frac{dN_{\bar{B}^0\rightarrow(f+\bar{f})}}{dt} = |\mathcal{A}_f|^2 e^{-\bar{\Gamma}t} \left\{ (1 + |\lambda|^2) \cosh(\Delta\gamma/2)t + \right. \\ \left. \text{Re}(\lambda^* + \kappa) \sinh(\Delta\gamma/2)t - \text{Im}(\lambda^* + \kappa) \sin(\Delta m)t \right\}. \end{aligned} \quad (18)$$

Adding together Eqs. (17) and (18) (corresponding to an untagged sample having equal numbers of B^0 and \bar{B}^0) recovers Eq. (16).

References

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